LESSON 10: THE LAPLACE TRANSFORM

ET 438a Automatic Control Systems Technology

Learning Objectives

After this presentation you will be able to:

- Explain how the Laplace transform relates to the transient and sinusoidal responses of a system.
- Convert time functions into the Laplace domain.
- Use Laplace transforms to convert differential equations into algebraic equations.
- Take the Inverse Laplace transform and find the time response of a system.
- Use Initial and Final Value Theorems to find the steady-state response of a system.

The Laplace Transform

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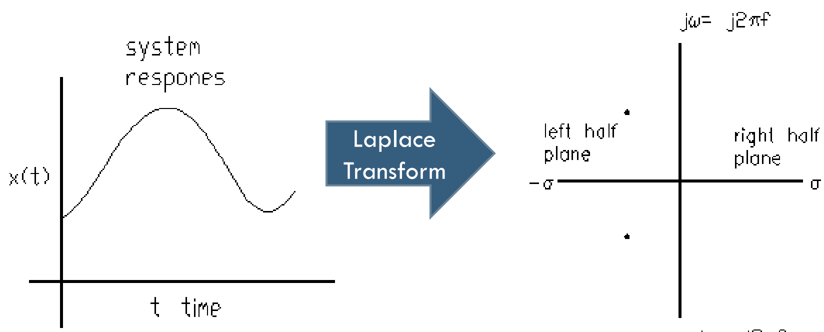
Laplace transform converts time domain problems into functions of a complex variable, s, that is related to the frequency response of the system



Time domain

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Complex frequency

Complex Frequency and The Laplace Transform

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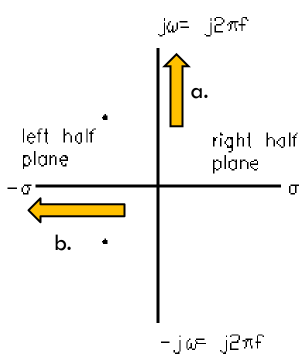

a.) Higher frequency
b.) Faster time constants

Complex frequency combines transient response with sinusoidal steady-state response to get total response of system to input

$s = \sigma + j\omega$ ← Complex Frequency

σ = exponential decay/increase constant that is related to time constants of systems transient response. RC = L/R = σ in circuit analysis

$e^{\sigma t}$ Exponentially increasing function over time

$e^{-\sigma t}$ Exponentially decreasing function over time

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Sinusoidal Response From Complex Frequency

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The radian frequency $j\omega = j2\pi f$ same frequency used in phasor analysis

Laplace related to sine response through Euler's Identity. Euler's relates complex exponentials to sine and cosine time functions

$$e^{j\omega t} = \cos(\omega t) + j\cdot\sin(\omega t)$$
$$e^{-j\omega t} = \cos(\omega t) - j\cdot\sin(\omega t)$$

Adding and subtracting the above relationships gives the exponential forms of sine and cosine

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Exponential Forms of Sinusoids

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Add the identities

$$e^{j\omega t} = \cos(\omega t) + j\cdot\sin(\omega t)$$
$$e^{-j\omega t} = \cos(\omega t) - j\cdot\sin(\omega t)$$

+

$$e^{j\omega t} + e^{-j\omega t} = 2\cdot\cos(\omega t)$$

$$\frac{e^{j\omega t} + e^{-j\omega t}}{2} = \cos(\omega t)$$

Exponential form of Cosine

Subtract the identities

$$e^{j\omega t} = \cos(\omega t) + j\cdot\sin(\omega t)$$
$$e^{-j\omega t} = \cos(\omega t) - j\cdot\sin(\omega t)$$

-

$$e^{j\omega t} - e^{-j\omega t} = 2j\cdot\sin(\omega t)$$

$$\frac{e^{j\omega t} - e^{-j\omega t}}{2j} = \sin(\omega t)$$

Exponential form of Sine

Since $e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}\cdot e^{j\omega t}$

Laplace can give complete response: dc transient and steady-state sinusoidal

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Basic Laplace Transform Pairs

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Time Domain Function | Laplace Domain Function

Time Domain Function		Laplace Domain Function
$\delta(t)$ Impulse	→	1
$u_s(t)$ Unit Step	→	$\frac{1}{s}$
e^{-at}	→	$\frac{1}{s+a}$
e^{at}	→	$\frac{1}{s-a}$
$\sin(\omega t)$	→	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	→	$\frac{s}{s^2+\omega^2}$
t Linear ramp (slope 1)	→	$\frac{1}{s^2}$

Note: time functions multiplied by constants give Laplace function multiplied by constant

Examples:

$$5\cdot u_s(t) \rightarrow \frac{5}{s}$$

$$3\cdot\sin(4t) \rightarrow 3\cdot\left(\frac{4}{s^2+16}\right) \quad \omega=4$$

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Laplace Transform Examples

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Match the following time functions to correct Laplace domain function using the transform pairs.

a.) $10t$

b.) $t\cdot e^{-at}$

c.) e^{5t}

d.) e^{-2t}

e.) $3\cdot\cos(t)$

Laplace table 3.2 textbook

1. $\frac{1}{s+2}$
2. $3\cdot\left(\frac{s}{(s^2+1)}\right)$
3. $\frac{1}{s-5}$
4. $\frac{10}{s^2}$
5. $\frac{1}{(s+a)^2}$

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Laplace Theorems

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Laplace of an unknown function

$$\mathcal{L}(f_1(t)) = F_1(s)$$

Capitalize unknown function name
Replace t with s

Laplace Operator Symbol

Examples

$$\mathcal{L}(i_1(t)) = I_1(s)$$

$$\mathcal{L}(v_1(t)) = V_1(s)$$

Linearity of transform - can multiply by constant

If $\mathcal{L}(f_1(t)) = F_1(s)$ and $\mathcal{L}(f_2(t)) = F_2(s)$

Then $\mathcal{L}(a \cdot f_1(t) + b \cdot f_2(t)) = a \cdot F_1(s) + b \cdot F_2(s)$

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Laplace Transforms of Calculus Operators

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Laplace Transform turns derivative into multiplication by s

If $\mathcal{L}(f_1(t)) = F_1(s)$

Then $\mathcal{L}\left(\frac{d}{dt} f_1(t)\right) = s \cdot F_1(s) - f_1(0)$

Subtract any non-zero initial conditions

For higher order derivatives

$$\mathcal{L}\left(\frac{d^2}{dt^2} f_1(t)\right) = s \cdot (s \cdot F_1(s) - f_1(0)) - \frac{d}{dt} f_1(0)$$

0 initial conditions reduces formula to

$$\mathcal{L}\left(\frac{d^2}{dt^2} f_1(t)\right) = s^2 \cdot F_1(s)$$

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Laplace Transforms of Calculus Operators

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Laplace turns integration into division by s

If $\mathcal{L}(f_1(t)) = F_1(s)$

Then $\mathcal{L}\left(\int f_1(t)\,dt\right) = \frac{1}{s}\cdot F_1(s)$

Examples from circuit analysis:
Capacitor voltage

$$v_C(t) = \frac{1}{C}\cdot\int i_C(t)\,dt$$

$$\mathcal{L}(v_C(t)) = \mathcal{L}\left(\frac{1}{C}\cdot\int i_C(t)\,dt\right)$$

$$V_C(s) = \frac{1}{C}\cdot\left(\frac{1}{s}\right)\cdot I_C(s) = \left[\frac{1}{C\cdot s}\right]\cdot I_C(s)$$

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More Examples From Circuit Analysis

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Find the Laplace relationship for inductor voltage

$$v_L(t) = L\cdot\frac{d}{dt}i_L(t)$$

$$\mathcal{L}(v_L(t)) = \mathcal{L}\left(L\cdot\frac{d}{dt}i_L(t)\right)$$

$$V_L(s) = L\cdot s\cdot I_L(s)$$

Laplace relationship for resistor voltage

$$v_R(t) = R\cdot i_R(t)$$

$$\mathcal{L}(v_R(t)) = \mathcal{L}(R\cdot i_R(t))$$

$$V_R(s) = R\cdot I_R(s)$$

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Laplace Transforms and Impedance

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Remember phasor analysis is only valid for sinusoidal steady-state. Turns ac analysis into an analysis similar to the dc. (Ohm's law)

Resistance	R	
Inductive Reactance	$X_L = j \cdot \omega \cdot L$	$\omega = 2\pi \cdot f \quad j = 90^\circ$
Capacitive Reactance	$X_C = \dfrac{1}{j \cdot \omega \cdot C} = -j \cdot \left(\dfrac{1}{\omega \cdot C}\right)$	$-j = \dfrac{1}{j} = -90^\circ$

Since Laplace variable represents frequency, it's possible to replace $j\omega$ with s and s with $j\omega$. If s is replaced with $j\omega$, analysis reverts to phasors We can find the frequency response of a dynamic system by converting differential equation into Laplace domain and replacing s with $j\omega$. Sweeping frequency produces Bode plot of system.

Laplace Transforms and Impedance

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Laplace "Impedances" (Ohm's Law)		Impedance (Phasors)	
Inductors	$Ls = \dfrac{V_L(s)}{I_L(s)}$	Inductors	$j\omega L = \dfrac{V_L(j\omega)}{I_L(j\omega)}$
Capacitors	$\dfrac{1}{Cs} = \dfrac{V_C(s)}{I_C(s)}$	Capacitors	$\dfrac{1}{j\omega C} = \dfrac{V_C(j\omega)}{I_C(j\omega)}$
Resistors	$R = \dfrac{V_R(s)}{I_R(s)}$	Resistors	$R = \dfrac{V_R(j\omega)}{I_R(j\omega)}$

Laplace Representations of OP AMP Circuits

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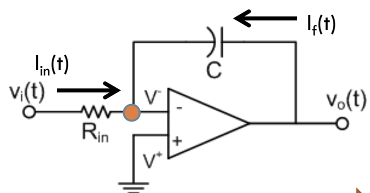
Find I/O relationship of integrator using Laplace relationships



Use OP AMP theory and solve. No I enters inverting node and $V^+=V^-=0$ due to ground connection.

Use KCL at inverting node

KCL

$$i_{in}(t) + i_f(t) = 0$$

$$i_{in}(t) = -i_f(t)$$

$$i_{in}(t) = \frac{V_{in}(t) - V^-(t)}{R_{in}} \qquad i_f(t) = C\frac{d}{dt}\left(V_o(t) - V^-(t)\right) \qquad V^+(t) = V^-(t) = 0 \quad \text{so}$$

Substitute into KCL equation

$$i_{in}(t) = \frac{V_{in}(t) - V(t)}{R_{in}} = \frac{V_{in}(t)}{R_{in}} \qquad i_f(t) = C\frac{d}{dt}\left(V_o(t) - V(t)\right) = C\frac{d}{dt}V_o(t) \qquad \frac{V_{in}(t)}{R_{in}} = -C\frac{d}{dt}V_o(t)$$

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Laplace Representations of OP AMP Circuits

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$$\frac{V_{in}(t)}{R_{in}} = -C\frac{d}{dt}V_o(t)$$

Integrate both sides of above equation to get $V_o(t)$. Integration is inverse of differentiation

$$\frac{1}{R_{in}}\int V_{in}(t)\,dt = -C\int \frac{d}{dt}V_o(t)\,dt$$

$$\frac{1}{R_{in}}\int V_{in}(t)\,dt = -CV_o(t)$$

$$\frac{-1}{R_{in}C}\int V_{in}(t)\,dt = V_o(t)$$

Take Laplace of Equation

$$-\frac{1}{R_{in}C}\left(\frac{1}{s}\right)V_{in}(s) = V_o(s)$$

Can use generalized gain formula of inverting OP AMP and Laplace Impedances

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Laplace Representations of OP AMP Circuits

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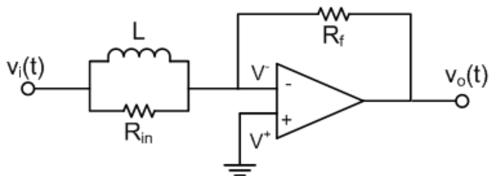
Example 10-1: Find the input/output relationship for the circuit shown below.



Generalized gain formula

$$\frac{V_o}{V_{in}} = \frac{-Z_f}{Z_{in}}$$

Use Laplace impedance relationships to find gain

$$\frac{V_o(s)}{V_{in}(s)} = \frac{-Z_f(s)}{Z_{in}(s)}$$

For inductor

$$V_L(t) = L\frac{d\,i_L(t)}{dt}$$

$$\mathcal{L}[V_L(t)] = V_L(s)$$

$$\mathcal{L}\left[L\frac{d\,i_L(t)}{dt}\right] = LsI_L(s)$$

$$\frac{V_L(s)}{I_L(s)} = Ls$$

so $Z_{in}(s) = R_{in} || Ls$

$$Z_{in}(s) = \frac{R_{in}(Ls)}{R_{in} + Ls}$$

$$Z_f(s) = R_f$$

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Example 10-1 Solution (2)

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Substitute into generalized gain formula

$$\frac{V_o(s)}{V_{in}(s)} = \frac{-R_f}{\frac{R_{in}Ls}{R_{in}+Ls}} = -R_f\left[\frac{R_{in}+Ls}{R_{in}Ls}\right] = \frac{-R_f}{R_{in}}\left[\frac{R_{in}+Ls}{Ls}\right]$$

$$\frac{V_o(s)}{V_{in}(s)} = \frac{-R_f}{R_{in}}\left[\frac{R_{in}}{Ls}+1\right] = \left(\frac{R_{in}}{L}\left(\frac{1}{s}\right)+1\right)\left(\frac{-R_f}{R_{in}}\right)$$

$\frac{R_{in}}{L}$ = time constant

$$V_o(s) = \left(\frac{R_{in}}{L}\left(\frac{1}{s}\right)+1\right)\left(\frac{-R_f}{R_{in}}\right)V_{in}(s)$$

$$-\left(\frac{R_f}{R_{in}}\right)\left(\frac{R_{in}}{L}\right)\left(\frac{1}{s}\right)V_{in}(s)$$

OUTPUT IS SUM OF CONSTANT GAIN

$$\frac{-R_f}{R_{in}}V_{in}(s)$$

And integrator action

$$-\frac{R_f}{Ls}V_{in}(s)$$

Division by s means Integration in time

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End Lesson 10: The Laplace Transform

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